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**Kinetic wealth exchange models:**
**Some analytical aspects**

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Abstract

We present some agent-based models aimed at explaining the statistical properties of wealth or income distribution. Theses models are inspired by Boltzmann’s kinetic theory of collisions in gases. We are particularly interested in a minimal model of closed economy where the agents exchange wealth amongst themselves such that the total wealth is conserved, and each individual agent saves a fraction \(0 \leq \lambda < 1\) of wealth before transaction. We show by moment calculations that the resulting wealth distribution cannot be the Gamma distribution that was conjectured in Phys. Rev. E 70, 016104 (2004). We also derive an upper bound for the distribution at low wealth which is a new result. A part of this work has been accepted for publication in 2010 [1].
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Chapter 1

Introduction

1.1 Econophysics

Before entering the main subject, we briefly introduce the young field of econophysics. The interaction between physics and economics is very old, for examples visionary suggestions by Majorana in the 30’s to use statistical physics in social science [2] or the explanation of the Brownian random walk by Louis Bachelier in his PhD thesis on the theory of speculation done 5 years before the Einstein’s works on diffusion [3], but econophysics in its institutionalized form came into existence only in 1995, when some papers on economic problems where published in physical journals. In these papers physicists (mainly specialists of statistical mechanics) try to tackle the complex problems posed by economics, especially by financial markets. Unsatisfied with the traditional explanations of economists, they “test a variety of new conceptual approaches deriving from the physical sciences” says H. Eugene Stanley who first coined the term “econophysics”. For a brief historical survey of this field see [4], for a review paper see [5] and for reference books see [6, 7]. Nowadays, econophysics is an exciting and rapidly growing field which offers a lot of challenging problems.

1.2 Wealth distribution

The distribution of wealth or income \(^1\) in society has been of great interest for many years. As first noticed by Pareto in the 1890’s [8], the wealth distribution seems to follow a “natural law” where the tail of the distribution is described by a power-law \(f(x) \sim x^{-(1+\alpha)}\).

\(^1\)Wealth is usually understood as things that have economic utility (monetary value or value of exchange), or material goods or property; it also represents the abundance of objects of value (or riches) and the state of having accumulated these objects; for our purpose, it is important to bear in mind that wealth can be measured in terms of money. Also income, defined as the amount of money or its equivalent received during a period of time in exchange for labor or services, from the sale of goods or property, or as profit from financial investments, is also a quantity which can be measured in terms of money. So both quantities can be represented by one variable.
The tail of the distribution describe typically the 5% of the population who have the higher wealth. Away from the tail, the distribution is better described by a Gamma or Log-normal distribution known as Gibrat’s law [9], see Fig. (1.1). Considerable investigation with real data during the last ten years revealed that the power-law tail exhibits a remarkable spatial and temporal stability and the Pareto index $\alpha$ is found to have a value between 1 and 2 [10, 11]. Even after 110 years the origin of the power-law tail remained unexplained but recent interest of physicists and mathematicians in econophysics has led to a new insight into this problem, see review [12] and references therein.

One of the current challenges is to write down the “microscopic equation” which governs the dynamics of the evolution of wealth distributions, possibly predicting the observed shape of wealth distributions.

The model of Gibrat [9] mentioned above and other models formulated in terms of a Langevin equation for a single wealth variable, subjected to multiplicative noise can lead to equilibrium wealth distributions with a power law tail, since they converge toward a log-normal distribution. However, the fit of real wealth distributions does not turn out to be as good as that obtained using e.g. a Gamma distribution.

Other models describe the wealth dynamics as a wealth flow due to exchanges between (pairs of) basic units. In this respect, such models are basically different from the class of models formulated in terms of a Langevin equation for a single wealth variable. For example, Levy and Solomon studied the generalized Lotka-Volterra equations in relation to power-law wealth distribution [13]. Ispolatov et al. [14] studied random exchange models of
wealth distributions. Other models describing wealth exchange have been formulated using matrix theory, the master equation, the Boltzmann equation or Markov chains approach.

In the following we will focus in a class of models referred to as kinetic wealth exchange models (KWEM) which are many-agent statistical models, where $N$ agents exchange a quantity $x$, defined as wealth or money. The name comes from the strong analogy between these models and the kinetic theory of gases where particles exchange energy. Depending on the exchange rule, the equilibrium wealth distribution can exhibit different shape. From various studies it is possible to predict power law distributions. However, a general understanding of the dependence of the shape of the equilibrium distribution on the underlying mechanisms and parameters is still missing.

During this internship the work was mainly focused on a particular KWEM: the global saving propensity model which will be presented later. In chapter 2 the main models are introduced and in chapter 3 results are presented and discussed, especially it is shown that the resulting wealth distribution of the global saving propensity model cannot be the Gamma distribution conjectured in [15].
Chapter 2

Kinetic wealth exchange models

For a long time physicists have been thinking that molecular models might give insight into the way empirical wealth distribution are linked to underlying dynamical processes that occur between the agents in society. Indeed as long as 1960 Mandelbrot said “There is a great temptation to consider the exchanges of money which occur in economic interaction as analogous to the exchanges of energy which occur in physics shocks between molecules” [16]. So let us start with the simplest of these models which is a very good first step to more complicated (and realistic) models.

2.1 Basic model

In the basic model, $N$ agents exchange a quantity $x$ which represents the wealth. The states of the system is characterized by the set $\{x_p\}$, $p = 1, 2, \ldots, N$, where $x_p$ is the wealth of agent $p$. The total wealth $E = \sum p x_p$ is conserved. The evolution of the system is then carried out according to a prescription, which defines the trading rule between agents. Dragulescu and Yakovenko introduced the following rule [17]: at every time step two agents $i$ and $j$ are extracted randomly, each one with same probability, and there is a random redistribution of the sum of the wealth of the two agents.

\[
x'_i = \epsilon (x_i + x_j), \\
x'_j = (1 - \epsilon) (x_i + x_j),
\]

where $x'_i$ and $x'_j$ are the agent wealths after the transaction has taken place. It can be noticed that in this way, the quantity $x$ is conserved during the single transactions: $x'_i + x'_j = x_i + x_j$. The interesting quantity is the wealth distribution $f(x)$ after a lot of time steps. $f$ is the probability density function i.e. $f(x) dx$ is the probability that in the steady state of the system, a randomly chosen agent will be found to have wealth between $x$ and $x + dx$. This wealth exchange looks like the kinetic energy exchange between two particles in a
ideal gases during an elastic collision. We have simulated this model and the resulting distribution is perfectly fitted by:

\[ f(x) = \frac{1}{\langle x \rangle} \exp \left( -\frac{x}{\langle x \rangle} \right). \]  

(2.2)

This equilibrium distribution of wealth is the same as the distribution of kinetic energy in a two dimensional gas with an effective temperature \( T = \langle x \rangle = E/N \) (see section (3.1)). The temperature is the average amount of wealth per agent which is the equipartition theorem in dimension two. This result can be demonstrated analytically by different methods: Boltzmann equation, entropy maximization, distributional equation, etc. Here I will present a geometrical derivation based on [18]. In the model, the total amount of money \( E \) is conserved,

\[ x_1 + x_2 + \cdots + x_{N-1} + x_N = E, \]

(2.3)

then this isolated system evolves on the positive part of an equilateral \( N \)-hyperplane. The surface area \( S_N(E) \) of an equilateral \( N \)-hyperplane of side \( E \) is given by

\[ S_N(E) = \frac{\sqrt{N}}{(N - 1)!} E^{N-1}. \]

(2.4)

If the ergodic hypothesis is assumed, each point on the \( N \)-hyperplane is equiprobable. Then the probability \( f(x_i)dx_i \) of finding agent \( i \) with money \( x_i \) is proportional to the surface area formed by all the points on the \( N \)-hyperplane having the \( i \)-th coordinate equal to \( x_i \). If the \( i \)-th agent has coordinate \( x_i \), the \( N-1 \) remaining agents share the money \( E - x_i \) on the \((N - 1)\)-hyperplane

\[ x_1 + x_2 + \cdots + x_{i-1} + x_{i+1} + \cdots + x_N = E - x_i, \]

(2.5)

whose surface area is \( S_{N-1}(E - x_i) \). It can be easily shown that

\[ S_N(E) = \sqrt{\frac{N}{N - 1}} \int_0^E S_{N-1}(E - x_i)dx_i. \]

(2.6)

Hence, we obtain

\[ f(x_i) = \sqrt{\frac{N}{N - 1}} \frac{S_{N-1}(E - x_i)}{S_N(E)}, \]

(2.7)

whose form after some calculation is

\[ f(x_i) = (N - 1)E^{-1} \left( 1 - \frac{x_i}{E} \right)^{N-2}. \]

(2.8)

Using \( \langle x \rangle \), the mean wealth per agent, we can write \( E = N \langle x \rangle \), then in the limit of large \( N \) we have

\[ \lim_{N \to \infty} \left( 1 - \frac{x_i}{E} \right)^{N-2} \simeq e^{-x_i/\langle x \rangle}. \]

(2.9)
The Boltzmann factor $e^{-x_i/\langle x \rangle}$ is found when $N \gg 1$ but, even for small $N$, it can be a good approximation for agents with low wealth. After normalization we obtain (2.2). This exponential distribution agrees with some data see [10] but in general it does not fit with real distributions. Thus, some improvement to this minimal model are required.

2.2 Global saving propensity

A step toward generalizing the basic model and making it more realistic, is the introduction of a saving criterion regulating the trading dynamics. Indeed, in a economic interaction agents do not want to put all their money at stakes, so they save a certain amount. This can be practically achieved by defining a saving propensity $0 \leq \lambda \leq 1$, which represents the fraction of wealth which is saved - and not reshuffled - during a transaction. The dynamics of the model is as follows [15]:

$$
\begin{align*}
    x'_i &= \lambda x_i + \epsilon (1 - \lambda) (x_i + x_j), \\
    x'_j &= \lambda x_j + (1 - \epsilon) (1 - \lambda) (x_i + x_j),
\end{align*}
$$

The conservation of money still holds, but the money which can be reassigned in a transaction between the $i$th and the $j$th agent has now decreased by a factor $(1 - \lambda)$. I have simulated this model for various values of $\lambda$, for a number of agents $N$ between 100 and $10^6$ and each agent having money 1 in the initial state. Again, at each time step the agents are are extracted randomly (each one with same probability). In each simulation a sufficient number of transactions was used to reach equilibrium. The numerical results are shown in Fig. 2.1.

The first observation is that the equilibrium distribution does not depend on the initial distribution or on the number of agents (as long as it is big enough $> 100$). This model leads to a qualitatively different equilibrium distribution than the basic one. In particular, it has a mode $x_m > 0$ and a zero limit for small $x$, i.e. $f(x) \xrightarrow{x\downarrow 0} 0$. The main challenge is to find an a closed-form expression of this distribution, the work done to answer to this question is presented in the following chapter. It is to be noticed that the saving propensity $\lambda$ introduced does not necessarily represent the agent’s investment strategy during a single wealth exchange. In fact stochastic processes provide in general a coarse grained description of the time evolution of a system: in the models considered here this means that one time step does not correspond to a single but rather to a large number of actual interactions between agents, the parameter $\lambda$ only modelling the fraction of wealth which on average an agent saves in the end of them. These transactions may be for instance wealth exchanges of various type, trades, or investments, which are influenced by many other parameters of the global system. The approximation which characterizes the type of models considered here is in the assumption that all these factors, when acting together and averaged over a large number of agent interactions, can be modelled by a single parameter.
Figure 2.1: Probability density for wealth $x$. The curve for $\lambda = 0$ is the Boltzmann-Gibbs function $f(x) = \langle x \rangle^{-1} \exp(-x/\langle x \rangle)$ for the basic model. The other curves correspond to a global saving propensity $\lambda > 0$. Solid lines: fit with function (3.9).
A natural extension of this model is then one with agents characterized by different saving propensities, as discussed in the following section.

### 2.3 Heterogeneous saving propensity

In a real economy the interest of saving varies from person to person. We move a step closer to the real situation where saving factor $\lambda$ is widely distributed within the population. The new trading rule read as follows:

$$
\begin{align*}
  x'_i &= \lambda_i x_i + \epsilon [(1 - \lambda_i) x_i + (1 - \lambda_j) x_j], \\
  x'_j &= \lambda_j x_j + (1 - \epsilon) [(1 - \lambda_i) x_i + (1 - \lambda_j) x_j],
\end{align*}
$$

(2.11)

where $\lambda_i$ and $\lambda_j$ are the saving propensities of agents $i$ and $j$. The agents have fixed (over time) saving propensities, distributed independently, randomly and uniformly within an interval 0 to 1. Starting with an arbitrary initial (uniform or random) distribution of money among the agents, the market evolves with the exchanges. The simulations are done with 500 agents and an average over the initial random assignment of the individual saving propensities is performed i.e. with a given configuration $\lambda_i$, the system is evolved until equilibrium is reached, then a new set of random saving propensities $\lambda'_i$ is extracted and reassigned to all agents, and the whole procedure is repeated many times. As a result of the average over the equilibrium distributions corresponding to the various $\lambda_i$ configurations, one obtains a distribution with a power law tail, $f(x) \sim x^{-a-1}$, see Fig. (2.2), where the Pareto exponent has the value $a = 1$. This value of the exponent has been predicted by various theoretical approaches [19, 20, 21]. This model is very interesting, however, in the following chapter we will focus exclusively on the global saving propensity model.
Figure 2.2: Probability density for wealth $x$ for heterogeneous saving propensity. The bottom plot is in log-log scale, the straight line indicates a power-law.
Chapter 3

Results and discussions

3.1 Distribution of kinetic energy in $D$ dimensions

Before entering into the analysis of the global model, we need a physical result: the distribution of the kinetic energy $K$ for a particle in an ideal isolated gas with energy $E$ in $D$ dimensions.

We start from a system Hamiltonian of the form

$$H(P,Q) = \frac{1}{2} \sum_{i=1}^{N} \frac{p_i^2}{m_i},$$

(3.1)

where $P = \{p_1, \ldots, p_N\}$ is the momentum vector of the $N$ particles. If we look at a particular particle the others particles act as a thermal bath at a temperature defined by $N k_B T/2 = E$. (In the following we will take $k_B = 1$.) Thus, this particle can be considered as a system in the canonical ensemble. Then, the normalized probability distribution in momentum space is simply

$$f(p) = \frac{1}{(2\pi mT)^{D/2}} \exp \left( -\frac{p^2}{2mT} \right),$$

(3.2)

where $p = (p_1, \ldots, p_D)$ is the momentum of a generic particle. It is convenient to introduce the momentum modulus $p$ of a particle in $D$ dimensions,

$$p^2 \equiv \mathbf{p}^2 = \sum_{k=1}^{D} p_k^2,$$

(3.3)

where the $p_k$'s are the Cartesian components, since the distribution (3.2) depends only on $p \equiv \sqrt{\mathbf{p}^2}$. One can then integrate the distribution over the $D - 1$ angular variables to
obtain the momentum modulus distribution function, with the help of the formula for the surface of a hypersphere of radius $p$ in $D$ dimensions,

$$S_D(p) = \frac{2\pi^{D/2}}{\Gamma(D/2)} p^{D-1}. \quad (3.4)$$

One obtains

$$f(p) = S_D(p) f(p) = \frac{2}{\Gamma(D/2)(2mT)^{D/2}} p^{D-1} \exp\left(-\frac{p^2}{2mT}\right). \quad (3.5)$$

The corresponding distribution for the kinetic energy $K = p^2/2m$ is therefore

$$f(K) = \left[\frac{dp}{dK} f(p)\right]_{p=\sqrt{2mK}} = \frac{1}{\Gamma(D/2)T} \left(\frac{K}{T}\right)^{D/2-1} \exp\left(-\frac{K}{T}\right). \quad (3.6)$$

### 3.2 Numerical analysis

We can see from the previous section that the basic model is analogous to an ideal gas in dimension two. Indeed, if you choose $D = 2$ in (3.6), you obtain

$$f(K) = 1 \exp\left(-\frac{K}{T}\right), \quad (3.7)$$

which is the same expression as (2.2), when the analogy $K \to x$ and $T \to \langle x \rangle$ is used. What is the effect of the saving propensity on this gas? What is the analogy when $\lambda \neq 0$? Maybe the ideal gas analogy holds but for another dimension, this idea is supported by the following analysis. With the saving propensity only a fraction of the wealth can be gained or lost in a transaction. Similarly in a gas, when the number of dimensions increases the head-on collisions, in which particles can exchange the total amount of energy are more and more rare. Thus, only a fraction of the total energy is actually gained or released on average in a collision. So, the analogy with an ideal gas can be extended in a dimension higher than 2 and increasing with $\lambda$. So the expression (3.6) might be used also for $\lambda \neq 0$, with $D > 2$.

Using the equipartition theorem :

$$T = \frac{2\langle K \rangle}{D}, \quad (3.8)$$

and rewriting (3.6) in terms of wealth variables, we have our fitting function :

$$f(x) = \frac{1}{\Gamma(n)} \left(\frac{n}{\langle x \rangle}\right)^n x^{n-1} \exp\left(-\frac{nx}{\langle x \rangle}\right). \quad (3.9)$$
Table 3.1: Analogy between kinetic and multi-agent model

<table>
<thead>
<tr>
<th>Exchanged quantity</th>
<th>Physical model</th>
<th>Economic model</th>
</tr>
</thead>
<tbody>
<tr>
<td>$K$</td>
<td>kinetic energy</td>
<td>$x =$ wealth</td>
</tr>
<tr>
<td>Units</td>
<td>$N$ particles</td>
<td>$N$ agents</td>
</tr>
<tr>
<td>Interaction</td>
<td>Collisions</td>
<td>Trades</td>
</tr>
<tr>
<td>Dimension</td>
<td>Integer $D$</td>
<td>Real number $2n(\lambda)$</td>
</tr>
<tr>
<td>Equipartition theorem</td>
<td>$k_B T = 2 \langle K \rangle / D$</td>
<td>$T(\lambda) = \langle x \rangle / n(\lambda)$</td>
</tr>
</tbody>
</table>

The fitting parameter is $n = D/2$, which we expect to be increasing with $\lambda$ and higher than 1 (because $D$ is higher than 2). All the curves for different $\lambda$ are extremely well-fitted by function (3.9) when $n$ is set to be:

$$n = 1 + \frac{3\lambda}{1 - \lambda} = 1 + \frac{2\lambda}{1 - \lambda}. \quad (3.10)$$

The parameter $n$ is found with the maximum likelihood estimator method and it is a real number greater than 1 and increasing with $\lambda$ (till $\infty$ when $\lambda = 1$). The difference with the gas is that the dimension is not necessary an integer. The analogy, summarized in Table 3.1, allowed us to find a fitting form which works very well.

This analogy is very elegant and seems intuitive but it is not a demonstration that the Gamma distribution (3.9) is the actual equilibrium distribution of the model. In the following we will present the analytical work done during the internship.

### 3.3 Fixed-point distribution

In order to obtain analytical results we are interested in the thermodynamic limit ($N \to \infty$). When the number of agents is very large, a particular agent will interact with another particular agent very rarely (because agents are chosen randomly with same probability). Thus, the agents can be considered independent, then at equilibrium, while the agents wealth follow the dynamics of Eq.(2.10), the global distribution does not change, so we can write:

$$X \overset{\text{d}}{=} \lambda X_1 + \epsilon(1 - \lambda)(X_1 + X_2), \quad (3.11)$$

where $\overset{\text{d}}{=} \text{ means identity in distribution and one assumes that the random variables } X_1, X_2 \text{ and } X \text{ have the same probability law, while the variables } X_1, X_2 \text{ and } \epsilon \text{ are stochastically independent.}$
3.3.1 Case $\lambda = 0$

Let us use Eq. (3.11) to confirm more formally the result (2.2) for the basic model. We set $\lambda = 0$ and hence we get

$$X \overset{d}{=} \epsilon(X_1 + X_2). \tag{3.12}$$

The question is, if $X_1$ and $X_2$ are distributed according to (2.2), what is the distribution of $X$. If it is the same, we have found the stationary distribution. So let us compute the distribution of $\epsilon(X_1 + X_2)$. Here we set $\langle x \rangle = 1$ for simplicity (but without loss of generality). The distribution of the sum of two random variables is the convolution of the individual distribution (see (A.1)):

$$f_{X_1+X_2}(x) = \int_0^x f_{X_1}(x') f_{X_2}(x-x') dx'$$

$$= \int_0^x e^{-(x-x')} e^{-x'} dx' = xe^{-x}, \tag{3.13}$$

$$\text{where } f_X(x) \text{ is the probability density function of the random variable } X. \text{ Then, we multiply this variable with a uniform distribution which leads to (see (A.2))}:

$$f_{\epsilon(X_1+X_2)}(x) = \int_x^{+\infty} \frac{1}{x} f_{X_1+X_2}(x') dx'$$

$$= \int_x^{+\infty} \frac{1}{x} xe^{-x'} dx' = e^{-x}. \tag{3.15}$$

We have showed that

$$f_X(x) = f_{X_1}(x). \tag{3.17}$$

So the distribution (2.2) is the equilibrium distribution for the basic model.

3.3.2 Case $\lambda \neq 0$

When $\lambda \neq 0$ it seems difficult to find the distribution of $X$, however, one can compute the moments of $f$. Indeed with (3.11), one can write immediately

$$\forall m \in \mathbb{N}, \langle X^m \rangle = \langle (\lambda X_1 + \epsilon(1-\lambda)(X_1 + X_2))^m \rangle, \tag{3.18}$$

by developing (3.18), one can find the recursive relation

$$\langle X^m \rangle = \sum_{k=0}^{m} \binom{m}{k} \frac{\lambda^{m-k}(1-\lambda)^k}{k+1} \sum_{p=0}^{k} \binom{k}{p} \langle X^{m-p} \rangle \langle X^p \rangle. \tag{3.19}$$
Using (3.19) with initial conditions $\langle X^0 \rangle = 1$ (normalization) and $\langle X^1 \rangle = 1$ (without loss of generality), we obtain

$$
\langle X^2 \rangle = \frac{\lambda + 2}{1 + 2\lambda}, \quad (3.20)
$$

$$
\langle X^3 \rangle = \frac{3(\lambda + 2)}{(1 + 2\lambda)^2}, \quad (3.21)
$$

$$
\langle X^4 \rangle = \frac{72 + 12\lambda - 2\lambda^2 + 9\lambda^3 - \lambda^5}{(1 + 2\lambda)^3(3 + 6\lambda - \lambda^2 + 2\lambda^3)}. \quad (3.22)
$$

Now let us compare these moments with conjecture (3.9)’s moments. Setting $\langle x \rangle = 1$ in Eq. (3.9) it is easy to show

$$
\langle x^k \rangle = \frac{(n + k - 1)(n + k - 2)...(n + 1)}{n^{k-1}}, \quad (3.23)
$$

writing (3.23) for $k = 2, 3, 4$ and choosing $n$ as in (3.9) we find

$$
\langle x^2 \rangle = \frac{n + 1}{n} = \frac{\lambda + 2}{1 + 2\lambda}, \quad (3.24)
$$

$$
\langle x^3 \rangle = \frac{(n + 2)(n + 1)}{n^2} = \frac{3(\lambda + 2)}{(1 + 2\lambda)^2}, \quad (3.25)
$$

$$
\langle x^4 \rangle = \frac{(n + 3)(n + 2)(n + 1)}{n^3} = \frac{3(\lambda + 2)(4 - \lambda)}{(1 + 2\lambda)^3}. \quad (3.26)
$$

The fourth moments are different so the conjecture that the Gamma distribution is an equilibrium solution of this model is wrong, nevertheless the first three moments coincide which shows that the Gamma-distribution is strangely a very good approximation. Moreover the deviation in the fourth moment is very small see Fig. 3.1. Finding a function that would coincide to higher moments is still an open challenge. These results are consistent with the ones found by Repetowicz et al. [19].

### 3.4 Upper bound form at low wealth range

Now, we know that the Gamma-distribution (3.9) can not be the stationary distribution. Finding the closed-form of the distribution is a huge challenge, so we first tried to find a form for small $x$. Unfortunately, we were just able to find an upper bound.

Starting with equation (3.11), we have for all $x \geq 0$

$$
P[X \leq x] = P[\lambda X_i + \epsilon(1 - \lambda)(X_i + X_j) \leq x]. \quad (3.27)
$$

For independent agents, we have

$$
\int_0^x dx f(x) = \int_0^\infty dx_i f(x_i) \int_0^\infty dx_j f(x_j) \int_0^1 d\Theta [x - \lambda x_i + \epsilon(1 - \lambda)(x_i + x_j)], \quad (3.28)
$$
where $\Theta$ is the Heaviside step function. Taking the derivative with respect to $x$ in both sides, we have

$$f(x) = \int_0^\infty dx_i f(x_i) \int_0^\infty dx_j f(x_j) \int_0^1 d\epsilon \delta[x - \lambda x_i - \epsilon (1 - \lambda)(x_i + x_j)].$$

(3.29)

This equation is an integral equation for $f(x)$. We are not able to solve it in closed form. However, one can simplify the equation, by doing the integral over $\epsilon$. Then the $\delta$-function will contribute only if we have the following constraints

$$0 \leq x_i \leq x/\lambda,$$

(3.30)

$$\frac{x - x_i}{1 - \lambda} \leq x_j,$$

(3.31)

$$0 \leq x_j.$$

(3.32)

The range defined by these constraints is shown in figure 3.2. In this range, the derivative of the argument of the delta function with respect to $\epsilon$ is just $(x_i + x_j)(1 - \lambda)$. And, hence we get

$$f(x) = \frac{1}{1 - \lambda} \int_0^{x/\lambda} dx_i f(x_i) \int_0^\infty dx_j f(x_j) \frac{1}{x_i + x_j}.$$

(3.33)

This immediately gives
Figure 3.2: Region of integration

\[ f(x) \leq C \int_0^{x/\lambda} f(x_i) dx_i, \quad (3.34) \]

where

\[ C = \frac{1}{1-\lambda} \int_0^\infty dx_j f(x_j) \frac{1}{x_j}. \quad (3.35) \]

We assume that \( f \) decays fast enough near 0 so that the integral in (3.35) is well defined. Now (3.34) may be rewritten as

\[ f(\lambda x) \leq C \int_0^x dx_i f(x_i). \quad (3.36) \]

We now use the observation that the numerically determined \( f(x) \) is a continuous function with a single maximum, say at \( x_0 \). Then for all \( x \leq x_0 \), the integrand (3.36) takes its maximum value at the right extreme point, i.e. when \( x_i = x \). This then gives us

\[ f(\lambda x) \leq C x f(x), \quad \text{for} \quad x \leq x_0. \quad (3.37) \]

Iterating this equation, we get

\[ f(\lambda^r x) \leq C^r x^{(r-1)/2} x^r f(x). \quad (3.38) \]

We can set \( x = x_0 \) in the above equation, giving

\[ f(\lambda^r x_0) \leq C^r x^{(r-1)/2} x_0^r f(x_0). \quad (3.39) \]
Then taking \( r \approx -\log x \) and rescaling the variables, we get

\[
 f(x) = O \left( x^\alpha \exp\left[ -\beta (\log x)^2 \right] \right), \tag{3.40}
\]

as \( x \to 0 \), where \( \alpha \) and \( \beta > 0 \) are two constants dependent on \( \lambda \). The Gamma-distribution decays slower than the rhs in (3.40) when \( x \to 0 \). This confirms again that the distribution of the global saving propensity model is not a Gamma-distribution.
Chapter 4

Conclusion and outlook

In this report we have presented several models of closed-economy aimed at explaining statistical regularities in wealth distributions. These models inspired by the kinetic theory of gas are very simple and reproduce the main features of empirical distributions. We were mainly focused on the global saving propensity model which reproduce very well real wealth distribution except the power-law tail. To reproduce the power-law tail an heterogeneous saving propensity is required. Nevertheless the global model is interesting because some aspects can be tackled analytically. Despite a lot of efforts we were not able to find an analytical form for the stationary distribution of this model but we have computed the exact moments of this distribution thanks to a new approach: the distributional equation. We showed that these moments agree with a previous conjecture [15] up to the third moment but start to deviate after, so the conjecture cannot be the actual distribution. We have also used the distributional equation (3.11) to show in a simple way that the Boltzmann factor is the stationary distribution for the basic model. We have found the form an upper-bound at low wealth which is a new result. Finding a closed-form for the equilibrium distribution is still an open challenge. Besides the KWEM presented here some interesting modifications have been introduced recently in the literature. We can mention Basu and Mohanty who tried to capture the growing nature of the market [22] or Basseti and Toscani who proposed the replace pointwise conservative exchanges by meanwise conservative exchanges [23]. We would like to point out that even tough KWEM have been the subject of intensive investigations they are still very challenging because very few analytical results are available. Finally, we can underline the fact that despite their simplicity, these kinetic models for wealth distributions seem to succeed in capturing the essential features of real wealth distributions, and provide toy systems that are as such interesting for statistical physics, but would hardly be written down from a pure physics perspective.
Appendix A

Two mathematical results on random variables

A.1 Sum of two independent random positive variables

Let \( X, Y \) be two independent random positive variables, with probability density function \( f_X \) and \( f_Y \). We define \( Z = X + Y \), with probability density function \( f_Z \). We want to express \( f_Z \) in terms of \( f_X \) and \( f_Y \). By definition, we have:

\[
P[Z \leq z] = P[X + Y \leq z], \tag{A.1}
\]

where \( P[.] \) means the probability of the event inside the brackets. \( X \) and \( Y \) are independent, then

\[
\int_0^z f_Z(z')dz' = \int_0^\infty dx f_X(x) \int_0^\infty dy f_Y(y) \Theta [z - (x + y)], \tag{A.2}
\]

where \( \Theta \) is the Heaviside step function. Taking the derivative with respect to \( z \) in both sides, we have

\[
f_Z(z) = \int_0^\infty dx f_X(x) \int_0^\infty dy f_Y(y) \delta [z - (x + y)] \tag{A.3}
\]

The argument is the delta function is equal to zero when \( y = z - x \) and \( x \leq z \), thus, we can write

\[
f_Z(z) = \int_0^z dx f_X(x) f_Y(z - x). \tag{A.4}
\]

A.2 Product of a positive random variable with a uniformly distributed variable

Let \( X \) be a random positive random variables with probability density function \( f_X \) and \( Y \) be a random variable uniformly distributed between 0 and 1. We define \( Z \) as: \( Z = XY \),

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with pdf $f_Z$. We want to express $f_Z$ in terms of $f_X$. By definition, we have:

$$
P[Z \leq z] = P[XY \leq z], \quad (A.5)
$$

$$
= P[X \leq \frac{z}{y}] \quad (A.6)
$$

$$
= \int_0^1 F_X \left( \frac{z}{y} \right) dy, \quad (A.7)
$$

where $F_X$ is the cumulative distribution function of $X$. Taking the derivative with respect to $z$ in both sides, we have

$$
f_Z(z) = \int_0^1 \frac{1}{y} f_X \left( \frac{z}{y} \right) dy. \quad (A.8)
$$

Using the change of variable $u = z/y$, we have

$$
f_Z(z) = \int_y^{\infty} \frac{1}{u} f_X(u) du. \quad (A.9)
$$
Bibliography


